

Inequalities for Weighted Polynomials

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IN MEMORY OF GÉZA FREUD

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $q_n(x)$ be a polynomial of degree n . The well-known inequality of Markov states that

$$\|q'_n(x)\|_{L_\infty[-1,1]} \leq n^2 \|q_n(x)\|_{L_\infty[-1,1]}$$

(cf. Timan [1, p. 218]). It is natural to conjecture that similar inequalities may hold for weighted polynomials on unbounded intervals, and indeed Milne [2] obtained a result which by a trivial change of variable is equivalent to the following inequality:

$$\|[\exp(-x^2/2) q_n(x)]'\|_{L_\infty(-\infty, \infty)} \leq K_1 n^{1/2} \|\exp(-x^2/2) q_n(x)\|_{L_\infty(-\infty, \infty)}, \quad (1)$$

where K_1 is a constant independent of n and of $q_n(x)$. Almost 40 years after the publication of Milne's paper, G. Freud showed that for $1 \leq p \leq \infty$,

$$\|\exp(-x^2/2) q'_n(x)\|_{L_p(-\infty, \infty)} \leq K_2 n^{1/2} \|\exp(-x^2/2) q_n(x)\|_{L_p(-\infty, \infty)} \quad (2)$$

(cf. [3, Theorem 1], and

$$\|\exp(-x^2/2) q_n(x)\|_{L_1(-\infty, \infty)} \leq K_3 \int_{-4\sqrt{n}}^{4\sqrt{n}} \exp(-x^2/2) |q_n(x)| dx \quad (3)$$

(cf. [3, Lemma 1]). Using (3), he also showed that

$$\|x \exp(-x^2/2) q_n(x)\|_{L_1(-\infty, \infty)} \leq K_4 n^{1/2} \|\exp(-x^2/2) q_n(x)\|_{L_1(-\infty, \infty)} \quad (4)$$

(cf. [3; Lemma 2]).

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It is easy to show that (2) and (4) imply (1); thus Freud's research can be considered to be a generalization of Milne's, although the former was apparently unaware of the work of the latter.

The purpose of this paper is to prove the following propositions:

THEOREM 1. (a) *Let $0 < r < \infty$, $1 \leq p \leq \infty$, and assume that $n > 0$ is an integer. Then there is a constant a , independent of n , such that*

$$\| |x|^r \exp(-x^2/2) q_n(x) \|_{L_p(-\infty, \infty)} \leq a n^{r/2} \| \exp(-x^2/2) q_n(x) \|_{L_p(-\infty, \infty)}. \quad (5)$$

(b) *The above inequality is optimal in the sense that the constant a cannot be replaced by a sequence $\{A_n\}$ that converges to zero as n tends to infinity.*

THEOREM 2. (a) *Let $s > 0$ and $n > 0$ be integers, and assume that $1 \leq p \leq \infty$. Then there is a constant b , independent of n , such that*

$$\| [\exp(-x^2/2) q_n(x)]^{(s)} \|_{L_p(-\infty, \infty)} \leq b n^{s/2} \| \exp(-x^2/2) q_n(x) \|_{L_p(-\infty, \infty)}. \quad (6)$$

(b) *The above inequality is optimal (in the sense of Theorem 1(b)).*

THEOREM 3. *Let $1 \leq p, p_1 \leq \infty$, and assume that $n > 0$ is an integer. Then there is a constant c , independent of n , p , and p_1 , such that*

$$\| \exp(-x^2/2) q_n(x) \|_{L_p(-\infty, \infty)} \leq c n^{11/(2p) - 1/(2p_1)} \| \exp(-x^2/2) q_n(x) \|_{L_{p_1}(-\infty, \infty)}. \quad (7)$$

THEOREM 4. *Let $1 \leq p \leq \infty$, and assume that $n > 0$ is an integer. Then there is a constant d , independent of n and p , such that*

$$\| \exp(-x^2/2) q_n(x) \|_{L_p(-\infty, \infty)} \leq d \| \exp(-x^2/2) q_n(x) \|_{L_p(-4\sqrt{n}, 4\sqrt{n})}. \quad (8)$$

Remarks. Note that (5), (6), and (8) generalize (4), (1), and (3), whereas (7) is similar to the results of Timan for polynomials and bounded intervals (see [1, p. 236, and also p. 229]). The case $p = \infty$ in Theorem 4 and some generalizations of it are known [4, 5].

2. PROOFS

Let $w(x) = \exp(-x^2/2)$, and let $\|\cdot\|_p$ stand for $\|\cdot\|_{L_p(-\infty, \infty)}$.

Proof of Theorem 1(a). We first show that if $q_n(x)$ is a polynomial of degree n , and r is nonnegative integer, there is a constant a_1 , such that

$$\|x^r w(x) q_n(x)\|_\infty \leq a_1 n^{r/2} \|w(x) q_n(x)\|_\infty. \quad (9)$$

Note that for $r = 0$, the assertion is trivial. Assume that (9) holds. Since

$$[w(x) x^r q_n(x)]' = w(x) [x^r q_n(x)]' - w(x) x^{r+1} q_n(x), \quad (10)$$

we have:

$$\|w(x) x^{r+1} q_n(x)\|_\infty \leq \| [w(x) x^r q_n(x)]' \|_\infty + \| w(x) [x^r q_n(x)]' \|_\infty,$$

and the proof of the inductive step follows by applying (1) and (9) to the first term in the right hand of the preceding inequality, and (2) and (9) to the second term.

We now show that (9) holds if r is any positive real number. To see this, note that for fixed n, q_n , and x , $|n^{-1/2} x|^r |w(x) q_n(x)|$ is a convex function of r ; it is thus readily seen that for fixed n and q_n , $h(r) = n^{-r/2} \| |x|^r w(x) q_n(x) \|_\infty / \| w(x) q_n(x) \|_\infty$ is convex on $[0, \infty)$. Thus if $r > 0$, and r_1 is an integer larger than r , we know that $h(r) \leq \max[h(0), h(r_1)]$, and the conclusion follows by noting that since (9) holds for integers, $h(0)$ and $h(r_1)$ are numbers independent of n and of $q_n(x)$.

We now prove (5). In view of (9), it suffices to assume that $1 \leq p < \infty$. We first need to prove Theorem 4 for $p < \infty$. From [3, p. 571, line 2] we see that if $|x| \geq 4\sqrt{n}$, and $1 \leq p < \infty$

$$|w(x) q_n(x)| \leq a_2 \|w(x) q_n(x)\|_1 \exp(-a_3 n), \quad (11)$$

where a_2 and a_3 are positive and independent of n . Thus,

$$\begin{aligned} & \int_{-n}^{-4\sqrt{n}} |w(x) q_n(x)|^p dx + \int_{4\sqrt{n}}^n |w(x) q_n(x)|^p dx \\ & \leq a_2^p (\|w(x) q_n(x)\|_1)^p \exp(-a_4 p n), \end{aligned}$$

where a_4 is positive and independent of n and p .

From [3, pp. 570–571, (3) and (5)] we infer that if $|x| \geq n$, then

$$|w(x) q_n(x)| \leq a_5 n \|w(x) q_n(x)\|_1 \exp(-x^2/16n).$$

Thus,

$$\begin{aligned} \int_{|x| \geq n} |w(x) q_n(x)|^p dx &\leq a_5^p n^p (\|w(x) q_n(x)\|_1)^p \int_n^\infty \exp(-px^2/16n) dx \\ &= a_5^p n^p (\|w(x) q_n(x)\|_1)^p \int_n^\infty x^{-1} [x \exp(-px^2/16n)] dx \\ &\leq a_5^p (\|w(x) q_n(x)\|_1)^p a_6 \exp(-a_7 pn), \end{aligned}$$

where a_5 , a_6 , and a_7 are independent of n and p . Setting $a_8 = \max(a_2, a_5 a_6, a_5)$, and $a_9 = \min(a_4, a_7)$, we conclude from the two preceding inequalities that

$$\left[\int_{|x| \geq 4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{1/p} \leq a_8 \|w(x) q_n(x)\|_1 \exp(-a_9 n). \quad (12)$$

From (3), dividing and multiplying by $1 + x^2$ and applying Hölder's inequality, we see that $\|w(x) q_n(x)\|_1 \leq \pi K_3 (1 + 16n) \|w(x) q_n(x)\|_p$. Since

$$\begin{aligned} \|w(x) q_n(x)\|_p &\leq \left[\int_{|x| \leq 4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{1/p} \\ &\quad + \left[\int_{|x| \geq 4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{1/p}, \end{aligned}$$

we conclude from (12) that there is a constant a_{10} , independent of n and p , such that

$$\|w(x) q_n(x)\|_p \leq a_{10} \left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{1/p}. \quad (13)$$

Assume now that r is an integer; thus $x^r q_n(x)$ is a polynomial of degree $n + r$, and from (13) we have

$$\begin{aligned} \|w(x) x^r q_n(x)\|_p &\leq a_{10} \left[\int_{-4\sqrt{n+r}}^{4\sqrt{n+r}} |w(x) x^r q_n(x)|^p dx \right]^{1/p} \\ &\leq a_{11} n^{r/2} \left[\int_{-4\sqrt{n+r}}^{4\sqrt{n+r}} |w(x) q_n(x)|^p dx \right]^{1/p} \\ &\leq a_{11} n^{r/2} \|w(x) q_n(x)\|_p, \end{aligned}$$

and (5) follows. We have therefore proved (5) for integral r .

To prove (5) for any $r \geq 0$, note that for fixed x , $[n^{-1/2} |x|^p]^r |w(x) q_n(x)|^p / (\|w(x) q_n(x)\|_p)^p$ is a convex function of r ; thus also Riemann sums of this function are convex, and we infer that

$n^{-r/2}(\| |x|^r w(x) q_n(x) \|_p)^p / (\| w(x) q_n(x) \|_p)^p$ is a convex function of r . The conclusion now follows as in the proof of (9).

(b) Assume $1 \leq p < \infty$, and let $q_n(x) = x^n$; thus

$$\begin{aligned} (\| |x|^r w(x) q_n(x) \|_p)^p &= \int_R |x|^{(r+n)p} \exp[-\frac{1}{2}(\sqrt{p}x)^2] dx \\ &= 2 \int_0^\infty |x|^{(r+n)p} \exp[-\frac{1}{2}(\sqrt{p}x)^2] dx \\ &= (2/p)^{(r+n)p+1/2} \int_0^\infty u^{(r+n)p-1/2} \exp(-u) du \\ &= (2/p)^{(r+n)p+1/2} \Gamma[\frac{1}{2}(rp + np + 1)]. \end{aligned}$$

Similarly, $(\| w(x) q_n(x) \|_p)^p = (2/p)^{(np+1)/2} \Gamma[\frac{1}{2}(np + 1)]$. Applying now Stirling's formula (cf. Lebedev [6, p. 12 (1.4.25)]), we see that $(\| |x|^r w(x) q_n(x) \|_p)^p \geq a_{12} n^{rp/2} (\| w(x) q_n(x) \|_p)^p$, and the conclusion follows. To prove the assertion for $p = \infty$, we use elementary calculus to conclude that $\| |x|^r w(x) q_n(x) \|_\infty = \exp[-\frac{1}{2}(n+r)](n+r)^{(n+r)/2}$, and $\| w(x) q_n \|_\infty = \exp[-(\frac{1}{2})n] n^{n/2}$, whence the conclusion readily follows. Q.E.D.

Proof of Theorem 2. (a) Assume first that $p = \infty$. We proceed by induction. For $s = 1$, (6) reduces to (1). Assume that (6) holds; applying (10), then (6), and finally (1), we obtain:

$$\begin{aligned} \| [w(x) q_n(x)]^{(s+1)} \|_\infty &= \| [w(x)(q'_n(x) - xq_n(x))]^{(s)} \|_\infty \\ &\leq b(n+1)^{s/2} \| w(x)[q'_n(x) - xq_n(x)] \|_\infty = b(n+1)^{s/2} \| [w(x) q_n(x)]' \|_\infty \\ &\leq bK_1(n+1)^{s/2} n^{1/2} \| w(x) q_n(x) \|_\infty \leq b_1 n^{(s+1)/2} \| w(x) q_n(x) \|_\infty, \end{aligned}$$

and the conclusion follows.

Assume now that $1 \leq p < \infty$. We again proceed by induction. The assertion is trivial for $s = 0$. Assume that (6) holds. Applying (10), (6), (2), and finally (4), we have:

$$\begin{aligned} \| [w(x) q_n(x)]^{(s+1)} \|_p &\leq \| [w(x) q'_n(x)]^{(s)} \|_p + \| [xw(x) q_n(x)]^{(s)} \|_p \\ &\leq bn^{s/2} \| w(x) q'_n(x) \|_p + b(n+1)^{s/2} \| xw(x) q_n(x) \|_p \\ &\leq bK_2 n^{(s+1)/2} \| w(x) q_n(x) \|_p + b_2 n^{s/2} \| xw(x) q_n(x) \|_p \\ &\leq b_3 n^{(s+1)/2} \| w(x) q_n(x) \|_p + b_2 K_4 n^{(s+1)/2} \| w(x) q_n(x) \|_p, \end{aligned}$$

and the conclusion follows.

(b) From the Rodrigues formula (cf. Szegő [7, p. 106, (5.5.3)]) and the chain rule, we readily infer that

$$w(x) H_n(x/\sqrt{2}) = (-1)^n 2^{n/2} [w(x)]^{(n)}.$$

Thus,

$$\begin{aligned} [w(x) H_n(x/\sqrt{2})]^{(s)} &= (-1)^n 2^{n/2} [w(x)]^{(n+s)} \\ &= (-1)^s 2^{-s/2} w(x) H_{n+s}(x/\sqrt{2}). \end{aligned} \quad (14)$$

From [4; p. 94 Exercise 6], we see by a change of variable that

$$\begin{aligned} \int_R |w(x) H_n(x/\sqrt{2})|^2 dx &= \int_R \exp(-x^2) H_n^2(x/\sqrt{2}) dx \\ &= \sqrt{2} \int_R \exp(-2x^2) H_n^2(x) dx = 2^{n-1/2} \Gamma(n+1/2), \end{aligned} \quad (15)$$

and

$$\int_R |w(x) H_{n+s}(x/\sqrt{2})|^2 dx = 2^{n+s-1/2} \Gamma(n+s+\frac{1}{2}).$$

Thus from (14) we see that

$$\int_R |[w(x) H_n(x/\sqrt{2})]^{(s)}|^2 dx = 2^{n-1/2} \Gamma(n+s+\frac{1}{2}); \quad (16)$$

combining (15) and (16) and applying Stirling's formula, we conclude that

$$\int_R |[w(x) H_n(x/\sqrt{2})]^{(s)}|^2 dx \geq b_3 n^s \int_R |w(x) H_n(x/\sqrt{2})|^2 dx,$$

i.e.,

$$\|[w(x) H_n(x/\sqrt{2})]^{(s)}\|_2 \geq b_4 n^{s/2} \|w(x) H_n(x/\sqrt{2})\|_2. \quad (17)$$

We have therefore proved the assertion for $p = 2$. To prove the assertion for all values of $p \geq 1$, we shall use the Riesz–Thorin interpolation theorem (cf. Zygmund [8, Vol. II, p. 95]; this method is briefly outlined in [3, p. 572]). Let $V_n(f; t)$ be defined as in Freud [9, p. 371, (7)]. As remarked in that paper, if $\gamma = [n/2]$, and if $f(x)$ is a polynomial of degree γ , then $V_n(f; t) = f(t)$ identically.

From [9, p. 371, (8) and (9)], and the Riesz–Thorin theorem, we readily infer that for all p such that $1 \leq p \leq \infty$, and every function $f(x)$ such that $w(x)f(x)$ is p -integrable,

$$\|w(x) V_n(f; x)\|_p \leq b_s \|w(x)f(x)\|_p. \quad (18)$$

Assume now that for some s and p there is a sequence M_n , converging to zero, such that for every polynomial $q_n(x)$ of degree n ,

$$\| [w(x) q_n(x)]^{(s)} \|_p \leq M_n n^{s/2} \|w(x) q_n(x)\|_p. \quad (19)$$

Define the linear operator $T_{n,s}(f)$ by

$$[T_{n,s}(f)](x) = [w(x) V_{2n}(f; x)]^{(s)}.$$

Let $M'_n = b_s M_n$; applying (19) and then (18) we have

$$\begin{aligned} \|T_{n,s}(f)\|_p &= \| [w(x) V_{2n}(f; x)]^{(s)} \|_p \leq M_{2n} (2n)^{s/2} \|w(x) V_{2n}(f; x)\|_p \\ &\leq M'_{2n} (2n)^{s/2} \|w(x) f(x)\|_p. \end{aligned} \quad (20)$$

If $p > 2$, let us choose a number q from the interval $(1, 2)$, whereas if $p < 2$, let $q > 2$; in either case, $\frac{1}{2} = \theta p^{-1} + (1 - \theta) q^{-1}$, where $0 < \theta < 1$. From (6) and (18),

$$\begin{aligned} \|T_{n,s}(f)\|_q &= \| [w(x) V_{2n}(f; x)]^{(s)} \|_q \leq b(2n)^{s/2} \|w(x) V_{2n}(f; x)\|_q \\ &\leq b_s (2n)^{s/2} \|w(x) f(x)\|_q. \end{aligned} \quad (21)$$

Applying (20), (21), and the Riesz–Thorin theorem, we thus conclude that $\|T_{n,s}(f)\|_2 \leq b_s^{1-\theta} (2n)^{s/2} (M'_{2n})^\theta \|w(x) f(x)\|_2$, for every function $f(x)$ such that $w(x)f(x)$ is square-integrable. In particular,

$$\begin{aligned} \| [w(x) H_n(x/\sqrt{2})]^{(s)} \|_2 &= \|T_{n,r}[H_n(x/\sqrt{2})]\|_2 \\ &\leq b_s^{1-\theta} (2n)^{s/2} (M'_{2n})^\theta \|w(x) H_n(x/\sqrt{2})\|_2. \end{aligned}$$

Since $(M'_{2n})^\theta$ converges to zero as $n \rightarrow \infty$ this contradicts (17), and the conclusion follows. Q.E.D.

Proof of Theorem 3. Assume that $p < p_1 < \infty$. Applying Pólya and Szegő [10, p. 65, Problem 71], with

$$\varphi(x) = |x|^{p_1/p} \quad \text{and} \quad f(x) = |w(x) q_n(x)|^p,$$

we see that

$$\left[(8\sqrt{n})^{-1} \int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{p_1/p} \leq (8\sqrt{n})^{-1} \int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^{p_1} dx$$

i.e.,

$$\left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^p dx \right]^{1/p} \leq (64n)^{(1/(2p) - 1/(2p_1))} \left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^{p_1} dx \right]^{1/p_1}.$$

Since clearly

$$\left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^{p_1} dx \right]^{1/p_1} \leq \|w(x) q_n(x)\|_{p_1}, \quad (22)$$

the conclusion readily follows from (13). If $p_1 = \infty$, the conclusion follows from (22) and (13) by noting that

$$\lim_{r \rightarrow \infty} \left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x) q_n(x)|^r dx \right]^{1/r} = \sup |w(x) q_n(x)|,$$

where the supremum is taken on $[-4\sqrt{n}, 4\sqrt{n}]$ (cf. e.g., Cotlar and Cignoli [11, p. 286, Lemma 1.2.3]).

The remainder of the proof is carried out by adapting an argument of Timan (cf. [1, p. 236]).

Assume $p_1 < p = \infty$. Let x_0 be such that $\|w(x) q_n(x)\|_{\infty} = |w(x_0) q_n(x_0)|$. Applying the mean value theorem and (6) we have:

$$\begin{aligned} |w(x_0) q_n(x_0)| - |w(x) q_n(x)| &\leq |w(x) q_n(x) - w(x_0) q_n(x_0)| \\ &\leq |x - x_0| \| [w(x) q_n(x)]' \|_{\infty} \leq bn^{1/2} |x - x_0| \|w(x) q_n(x)\|_{\infty}. \end{aligned}$$

In view of the definition of x_0 , it is therefore clear that for all real x

$$[1 - bn^{1/2} |x - x_0|] \|w(x) q_n(x)\|_{\infty} \leq |w(x) q_n(x)|. \quad (23)$$

Assume, e.g., that $x > x_0$, and let $b_n = x_0 + (bn^{1/2})^{-1}$; then

$$\int_{x_0}^{b_n} [1 - bn^{1/2} |x - x_0|]^{p_1} dx = [(bn^{1/2})(p_1 + 1)]^{-1}.$$

Raising both terms of (23) to the p_1 th power and integrating, we thus conclude that

$$[(bn^{1/2})(p_1 + 1)]^{-1} (\|w(x) q_n(x)\|_{\infty})^{p_1} \leq \int_{x_0}^{b_n} |w(x) q_n(x)|^{p_1} dx,$$

whence

$$\|w(x) q_n(x)\|_{\infty} \leq (p_1 + 1)^{1/p_1} (bn^{1/2})^{1/p_1} \|w(x) q_n(x)\|_{p_1},$$

i.e.,

$$\|w(x) q_n(x)\|_{\infty} \leq cn^{1/(2p_1)} \|w(x) q_n(x)\|_{p_1},$$

where the constant c is independent of n and p_1 .

If $p_1 < p < \infty$, we have

$$\begin{aligned} (\|w(x) q_n(x)\|_p)^p &= \int_R |w(x) q_n(x)|^p dx \\ &= \int_R |w(x) q_n(x)|^{p_1} |w(x) q_n(x)|^{p-p_1} dx \\ &\leq (\|w(x) q_n(x)\|_\infty)^{p-p_1} (\|w(x) q_n(x)\|_{p_1})^{p_1}, \end{aligned}$$

i.e.,

$$(\|w(x) q_n(x)\|_p)^p \leq (\|w(x) q_n(x)\|_\infty)^{p-p_1} (\|w(x) q_n(x)\|_{p_1})^{p_1}.$$

Thus

$$(\|w(x) q_n(x)\|_p)^p \leq (cn^{1/(2p_1)} \|w(x) q_n(x)\|_{p_1})^{p-p_1} (\|w(x) q_n(x)\|_{p_1})^{p_1},$$

whence the conclusion readily follows.

Q.E.D.

Proof of Theorem 4. For $p < \infty$, the conclusion follows from (13). Assume therefore that $p = \infty$, and select $a > 0$ and r , $1 < r < \infty$, arbitrarily. Since clearly

$$\|w(x) q_n(x)\|_{L_{r| -a, a|}} \leq \|w(x) q_n(x)\|_{L_r},$$

we infer from (13) that

$$\|w(x) q_n(x)\|_{L_{r| -a, a|}} \leq a_{10} \|w(x) q_n(x)\|_{L_{r| -4\sqrt{n}, 4\sqrt{n}|}}.$$

Since a_{10} does not depend on r , making $r \rightarrow \infty$ we see that

$$\|w(x) q_n(x)\|_{L_{\infty| -a, a|}} \leq a_{10} \|w(x) q_n(x)\|_{L_{\infty| -4\sqrt{n}, 4\sqrt{n}|}},$$

and the conclusion follows by noting that

$$\|w(x) q_n(x)\|_\infty = \sup\{\|w(x) q_n(x)\|_{L_{\infty| -a, a|}}, a > 0\}.$$

Q.E.D.

Remark. After writing this paper the author discovered that another proof of Theorem 2(a), with a more general weight function, was obtained by G. Freud in [12, p. 129, Theorem 2]. As part of a forthcoming article the author will include the generalization of results of this paper, for Freud's weight function.

REFERENCES

1. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," J. Berry, transl., Macmillan, New York, 1963.
2. W. E. MILNE, On the maximum absolute value of the derivative of $e^{-x^2}P_n(x)$, *Trans. Amer. Math. Soc.* **33** (1931), 143–146.
3. G. FREUD, On an inequality of Markov type, *Soviet Math. Dokl.* **12** (1971), 570–573. (Revised translation from *Dokl. Akad. Nauk SSSR* **197** (1971)).
4. G. FREUD, On two polynomial inequalities. I. *Acta Math. Acad. Sci. Hungar.* **22** (1971), 109–116.
5. G. P. NÉVAI, A certain inequality of G. Freud. (Russian), *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **16** (1973), 87–92.
6. N. N. LEBEDEV, "Special Functions and Their Applications," revised English ed. (R. A. Silverman, Ed. and transl.), Dover, New York, 1972.
7. G. SZEGÖ, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ. No. 23, Amer. Math. Soc., Providence, R. I., 1975.
8. A. ZYGMUND, "Trigonometric Series," Vol. I and II, 2nd ed., Cambridge Univ. Press, Cambridge, repr., 1968.
9. G. FREUD, On weighted approximation by polynomials on the real axis, *Soviet Math. Dokl.* **11** (1970), 370–372. (Translation from *Dokl. Akad. Nauk. SSSR* **191** (1970)).
10. G. PÓLYA AND G. SZEGÖ, "Problems and Theorems in Analysis," Vol. I, (D. Aeppli, transl.), Springer-Verlag, New York, 1976.
11. M. COTLAR AND N. CIGNOLI, "An Introduction to Functional Analysis," American Elsevier, New York, 1974.
12. G. FREUD, On direct and converse theorems in the theory of weighted polynomial approximation, *Math. Z.* **126** (1972), 123–134.